Completeness of the ZX-calculus

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Outline

Birth of the ZX-calculus

- It all started with the paper [Abramsky, Coecke, LICS'04] which initiated categorical quantum mechanics.
- The ZX-calculus made its debut in [Coecke, Duncan, ICALP'08]. It was not yet named ZX-calculus.
- The term <u>ZX-calculus</u> first appeared in the paper [Coecke, Duncan, New Journal of Physics, 2011].

The concept of ZX-calculus

The main objective of the ZX-calculus is to “describe” quantum theory starting with a pair of complementary quantum observables.

- Characterising their interactions;
- Adding “phases”;
- Expressing the equations as diagrams.

This captures some interesting aspects of quantum theory.

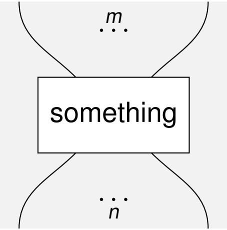
- Non-locality;
- Simple diagrammatic explanation.

Can we provide a complete equational axiomatisation of quantum theory??

Framework

(†-) self-dual, compact closed PROP. We will let the pictures do the talking…

- String diagrams



Composition of diagrams:

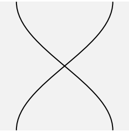
- Plugging
- Juxtaposition

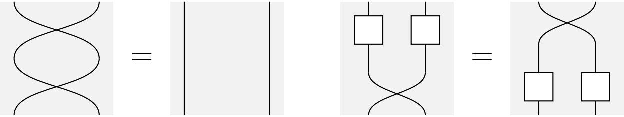
Framework

- Some special diagrams

Framework

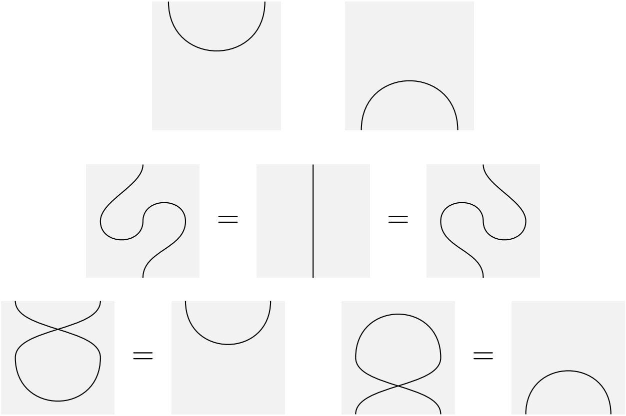
- Permuting wires





Framework

- Bending wires



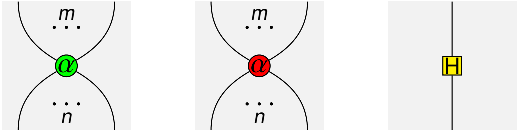
Framework

Bottom line is. . .

just bend and twist the wires and slide the boxes.

The language of ZX-calculus

Fill the boxes with spiders.

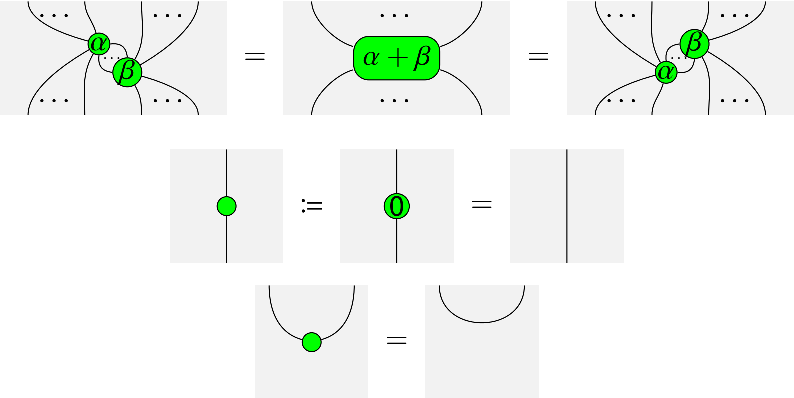


- These spiders correspond to complementary quantum observables;
- The yellow box changes between the observables;

We want to characterise their interactions.

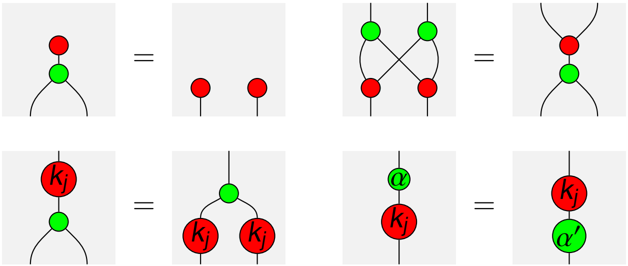
Typical axioms/rules of the ZX-calculus

Spider law



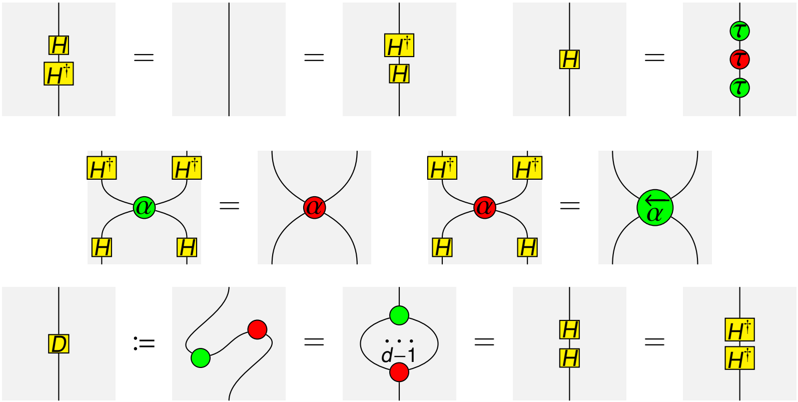
Typical axioms/rules of the ZX-calculus

Interaction



Typical axioms/rules of the ZX-calculus

Quantum Fourier Transform

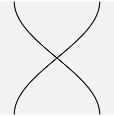


Interpreting the ZX-diagrams

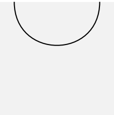
We can interpret the ZX-diagrams as some linear map in **FdHilb**.
For a fix Hilbert space dimension d,

$\mapsto \quad 1$

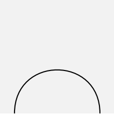
$\mapsto \quad \sum_{j=0}^{d-1} |j\rangle\langle j|$



$\mapsto \quad \sum_{i,j=0}^{d-1} |i\ j\rangle\langle j\ i|$

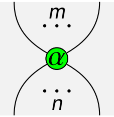


$\mapsto \quad \sum_{j=0}^{d-1} \langle j\ j|$

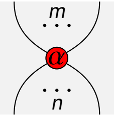


$\mapsto \quad \sum_{j=0}^{d-1} |j\ j\rangle$

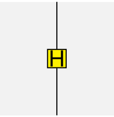
Interpreting the ZX-diagrams



$$\mapsto \quad \sum_{j=0}^{d-1} e^{i\alpha_j} |j \dots j\rangle\langle j \dots j|$$



$$\mapsto \quad \sum_{j=0}^{d-1} e^{i\alpha_j} \left|h_j \dots h_j\right\rangle\left\langle h_j \dots h_j\right|$$



$$\mapsto \quad \sum_{j=0}^{d-1} \left|h_j\right\rangle\langle j|$$

$$\alpha_0 = 0, \quad \left|h_j\right\rangle = \frac{1}{\sqrt{d}} \sum_{l=0}^{d-1} \omega^{jl} |l\rangle, \quad \omega = e^{i\frac{2\pi}{d}}$$

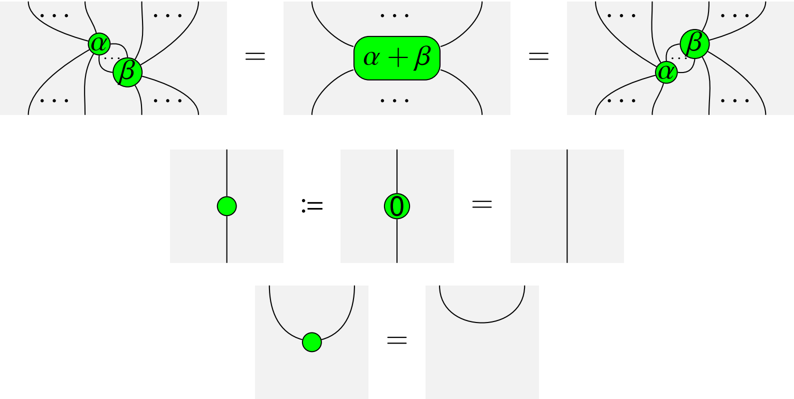
Interpreting the ZX-diagrams

A composite diagram is interpreted as the composition of the matrices.

- Plugging $\rightarrow$ matrix multiplication;
- Juxtaposition of diagrams $\rightarrow$ kronecker product.

Looking back at the rules

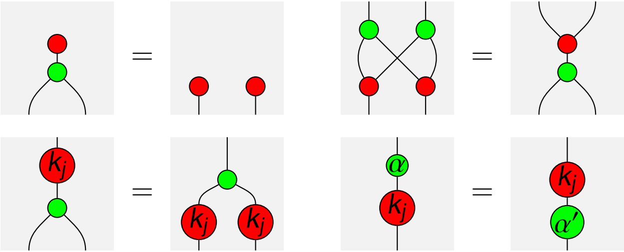
Spider law



$\alpha = (\alpha_1, \ldots, \alpha_{d-1})$ where $\alpha_j \in [0, 2\pi)$, and the addition is mod 2π.

Looking back at the rules

Interaction



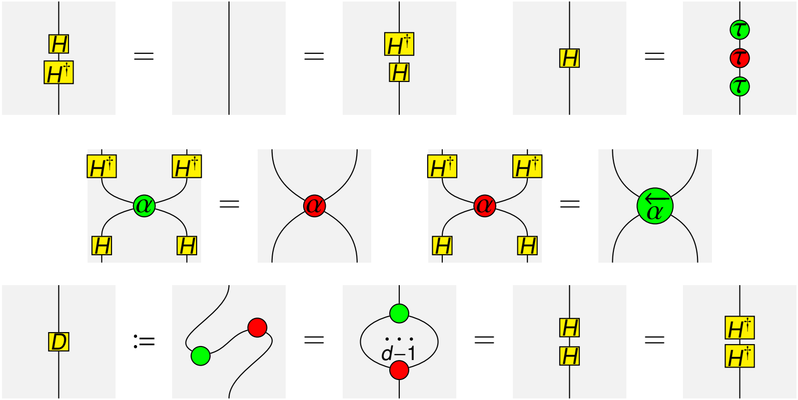
The k_j red nodes are permutation matrices which forms a cyclic group.

$$k_j = \left(j\frac{2\pi}{d}, 2j\frac{2\pi}{d}, \ldots, (d-1)\frac{2\pi}{d} \right)$$

$$\alpha' = \alpha_{1-j} - \alpha_{-j}, \cdots, \alpha_{d-1-j} - \alpha_{-j}$$

Looking back at the rules

Quantum Fourier Transform



$$\tau = (t_1, \ldots, t_{d-1}), \qquad t_j = j\pi + j^2 \frac{\pi}{d}$$
$$\overleftarrow{\alpha} = (\alpha_{d-1}, \ldots, \alpha_1)$$

Sound, Universal, Completeness

Definition (Sound, Universal, Complete)

- The interpretation is <u>sound</u> if any equal ZX-diagrams have equal interpretation.
- The interpretation is <u>universal</u> if for each linear map in $\mathbf{FdHilb}_d$, there exist a ZX-diagram that is interpreted as such.
- The interpretation is <u>complete</u> if every equal interpreted diagrams are equal diagrams.

In other words, the interpretation $[\![.]\!] : ZX \to \mathbf{FdHilb}_d$ is sound if it is functorial, it is universal if it is surjective on objects and morphisms, and is complete if it is faithful.
This interpretation is the standard interpretation we normally use.
We just say the ZX-calculus is sound, is universal, or is complete.

Sound, Universal, Completeness

To prove that the ZX-calculus is sound, we just have to check all the ZX rules under the standard interpretation are true equalities.

- The qudit ZX-calculus is sound for any $d \geq 2$ [Ranchin, QPL'14].

Universality is a bit tricky, but we also have a nice result.

- The qudit ZX-calculus is universal for any $d \geq 2$ [Wang and Bian, QPL'14].

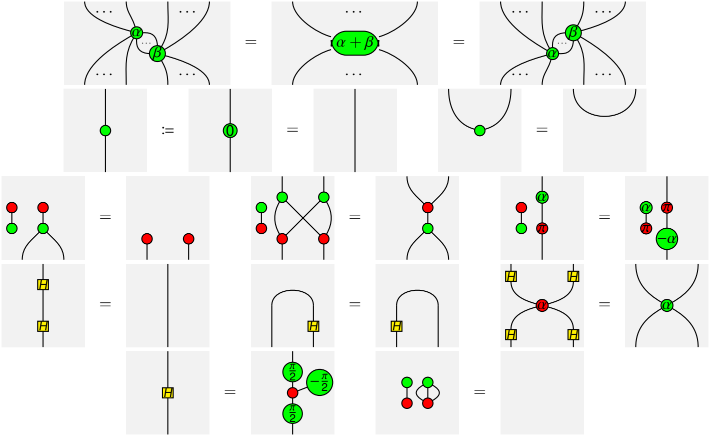
What about completeness result? No results yet for the general case, but there are numerous results when d = 2.

Qubit systems

Let us talk about the ZX-calculus for qubit systems.

- The spiders are still the same, only now they are decorated by one angle;
- Some of the rules are now redundant
  - Fourier transform $\rightarrow$ Hadamard gate, which is self adjoint;
  - Green and red cups and caps are the same;
  - Dualiser is just a straight wire;

Qubit ZX-calculus



Fragments of ZX-calculus

- Real stabilizer: multiple of π;
- Stabilizer: multiple of $\frac{\pi}{2}$;
- Clifford+T: multiple of $\frac{\pi}{4}$;
- Unrestricted (full): full range

Original rules of ZX-calculus

[Coecke, Duncan, New J. Phys., 2011]

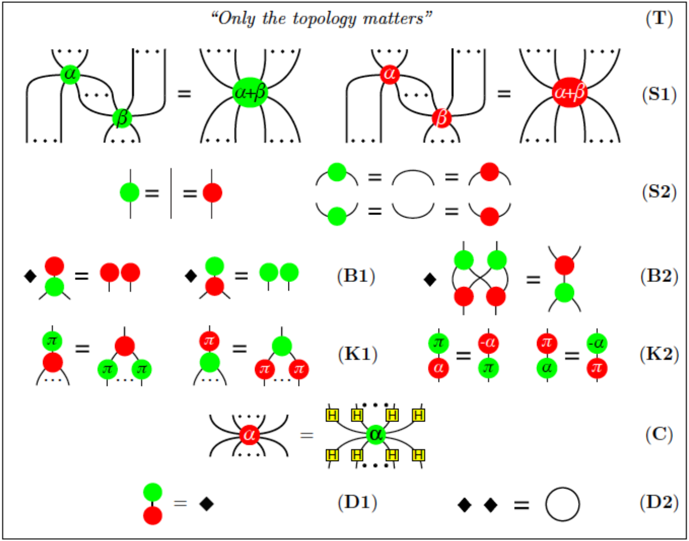
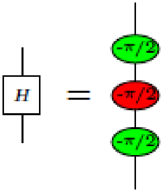


Figure 1. Rules for the zx-calculus

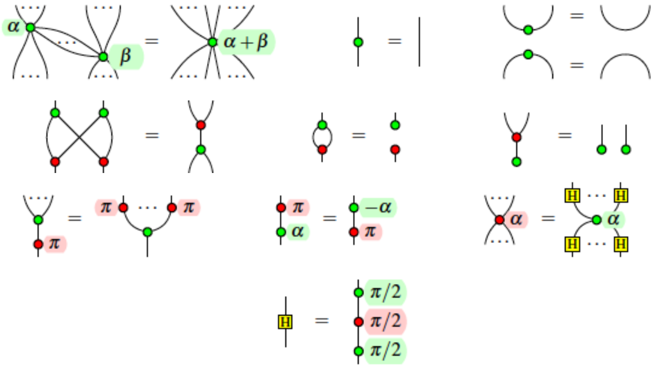
Necessity of Euler decomposition of Hadamard gate

[Duncan, Perdrix, CiE'09]
Euler decomposition of H cannot be derived from the original rules of ZX-calculus:



Completeness for stabilizer QM

[Backens, QPL'12, New J. Phys.'14]
The ZX-calculus is complete for stabilizer QM ($\pi/2$ fragment QM):



where $\alpha, \beta \in \{k\frac{\pi}{2} | k \in \mathbb{Z}\}$.

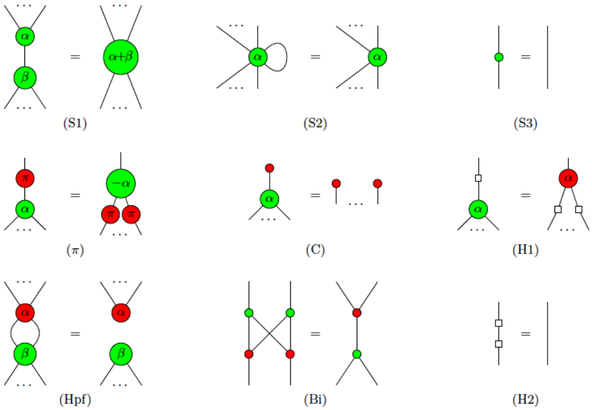
Ideas the on proof of the completeness for stabilizer QM

- Use map-state duality to turn linear maps (matrices) into states (vectors),
- Turn stabilizer states to graph states with local Clifford operators (GS-LC diagrams),
- Perform local complementation on GS-LC diagrams,
- Give ”quasi-normal form”: reduced GS-LC diagram. Compare pairs of simplified reduced GS-LC diagram.

Completeness for real stabilizer QM

[Duncan, Perdrix, QPL'13]

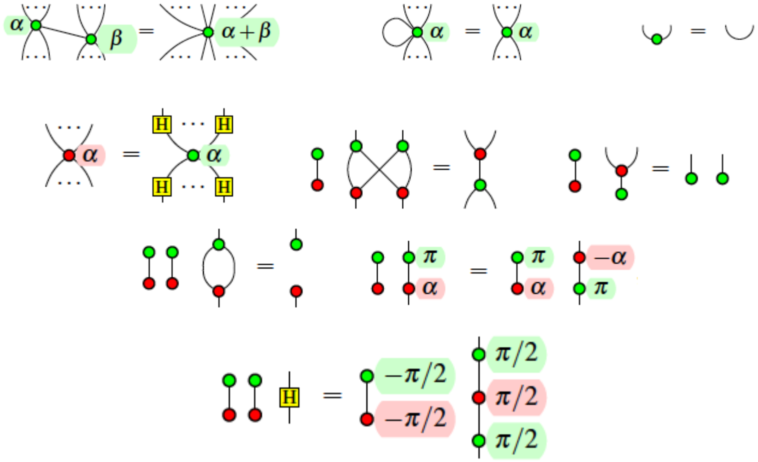
The ZX-calculus is complete for real stabilizer QM (π fragment QM):



where $\alpha, \beta \in \{0, \pi\}$. The proof is someone the same for the stabilizer QM.

Completeness for scalar stabilizer QM

[Backens, QPL'15]
The stabilizer ZX-calculus is complete for scalars:



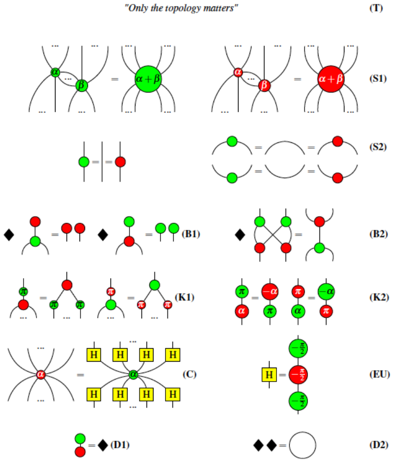
Ideas on the proof of completeness for scalar stabilizer QM

- Give normal form for non-zero stabilizer scalars and zero diagrams,
- Use completeness of scalar-free stabilizer ZX-calculus.

Incompleteness of ZX-calculus

[Schr$\ddot{o}$der, Zamdzhiev, QPL'14]

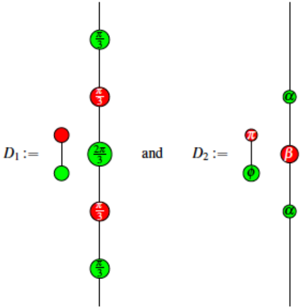
The following ZX rules are incomplete for the entire pure qubit QM:



where $\alpha, \beta \in [0, 2\pi)$.

Incompleteness of ZX-calculus

The following equality can not be derived from ZX rules shown in last slide:



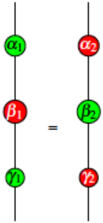
$$\alpha := -\arccos\left(\frac{5}{2\sqrt{13}}\right) \approx 0.2561\pi$$

$$\beta := -2\arcsin\left(\frac{\sqrt{3}}{4}\right) \approx -0.2851\pi$$

$$\phi := \arcsin\left(\frac{\sqrt{3}}{4}\right) - \alpha \approx 0.3987\pi$$

Conjecture on completeness of ZX-calculus

we believe that a "color-swap" rule of the form:



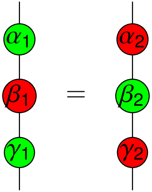
might be needed. This would require identifying functions f_1, f_2, f_3, s.t. the above rule is valid and

$$\alpha_1 = f_1(\alpha_2, \beta_2, \gamma_2)$$
$$\beta_1 = f_2(\alpha_2, \beta_2, \gamma_2)$$
$$\gamma_1 = f_3(\alpha_2, \beta_2, \gamma_2)$$

In other words, an analytic solution for converting from ZXZ to XZX Euler decompositions of single-qubit unitary gates is required.

Analytic solution for permuting colours

[Coecke, Wang, RC'18]



with

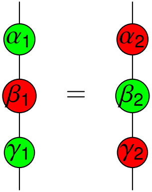
$$\begin{cases} \alpha_2 = \arg z + \arg z_1 \\ \beta_2 = 2\arg(|\frac{z}{z_1}| + i) \\ \gamma_2 = \arg z - \arg z_1 \end{cases}$$

where:

$$z = \cos\frac{\beta_1}{2}\cos\frac{\alpha_1+\gamma_1}{2} + i\sin\frac{\beta_1}{2}\cos\frac{\alpha_1-\gamma_1}{2}$$
$$z_1 = \cos\frac{\beta_1}{2}\sin\frac{\alpha_1+\gamma_1}{2} - i\sin\frac{\beta_1}{2}\sin\frac{\alpha_1-\gamma_1}{2}$$

Analytic solution for permuting colours

[Coecke, Wang, RC'18]



with
$$\begin{cases} \alpha_2 = \arg z + \arg z_1 \\ \beta_2 = 2\arg(|\frac{z}{z_1}| + i) \\ \gamma_2 = \arg z - \arg z_1 \end{cases}$$

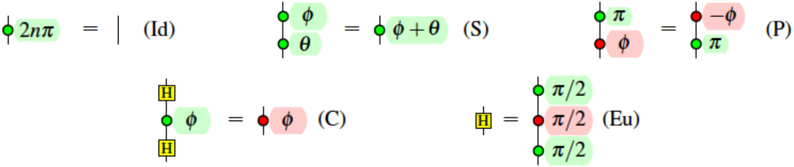
where:

$$z = \cos\frac{\beta_1}{2}\cos\frac{\alpha_1+\gamma_1}{2} + i\sin\frac{\beta_1}{2}\cos\frac{\alpha_1-\gamma_1}{2}$$
$$z_1 = \cos\frac{\beta_1}{2}\sin\frac{\alpha_1+\gamma_1}{2} - i\sin\frac{\beta_1}{2}\sin\frac{\alpha_1-\gamma_1}{2}$$

Remark: We call this as (P) rule rather than (EU) rule, because it is not about Euler decomposition of a unitary, but rather on how to (P)ermute colours given an Euler decomposition. Also there is already a well-known ZX rule called (EU).

Completeness for single-qubit Clifford+T circuits

[Backens, QPL'14]
The ZX-calculus is complete for single-qubit Clifford+T ($\pi/4$ fragment) circuits:



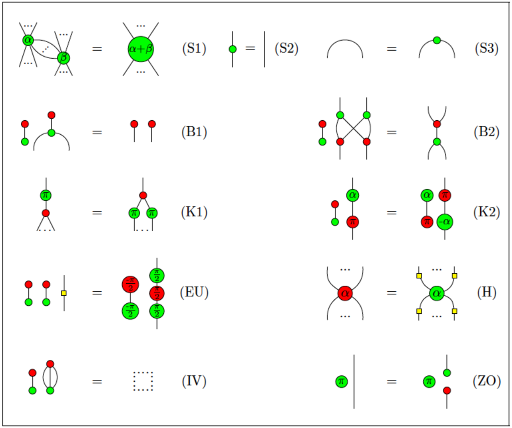
where $n \in \mathbb{Z}$, and $\theta, \phi \in \{k\frac{\pi}{4} | k \in \mathbb{Z}\}$.

Ideas on the proof of the completeness for single-qubit Clifford+T circuits

- Normal form, . . . , and lots of technical stuff. . .

Incompleteness for multiple-qubit Clifford+T QM

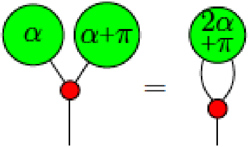
[Perdrix, Wang, MFCS'16]
The following ZX rules are incomplete for multiple-qubit Clifford+T ($\pi/4$ fragment) QM:



where $\alpha, \beta \in \{k\frac{\pi}{4} | k \in \mathbb{Z}\}$.

Necessity of supplementarity

[Perdrix, Wang, MFCS'16]
The following equality called supplementarity cannot be derived if $\alpha \not\equiv 0 \bmod \frac{\pi}{2}$:



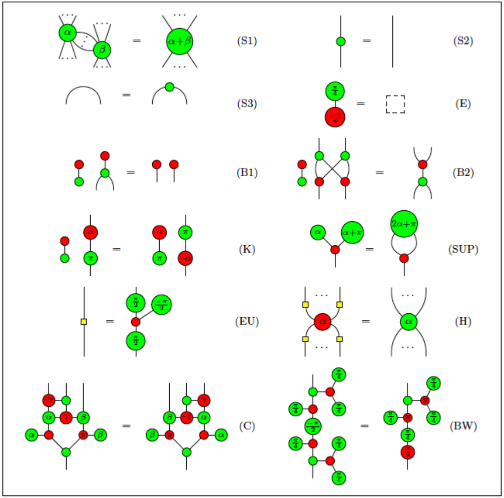
Completeness?

The chase continued, until...

The first completeness result of Clifford+T ZX-calculus

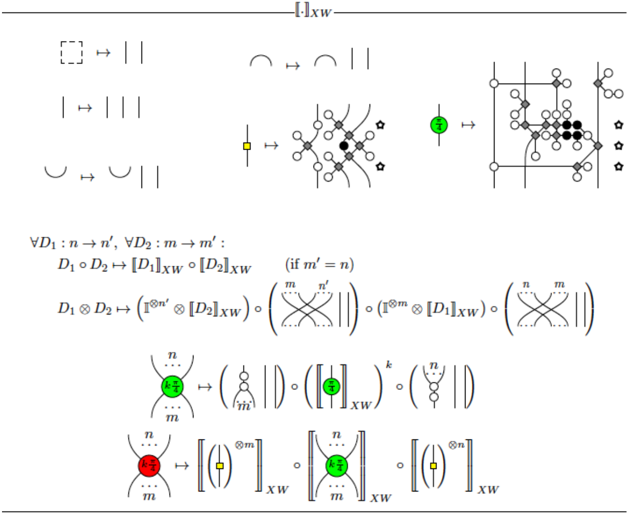
[Jeandel, Perdrix, and Vilmart, LICS'18]

The following ZX rules are complete for Clifford+T QM:



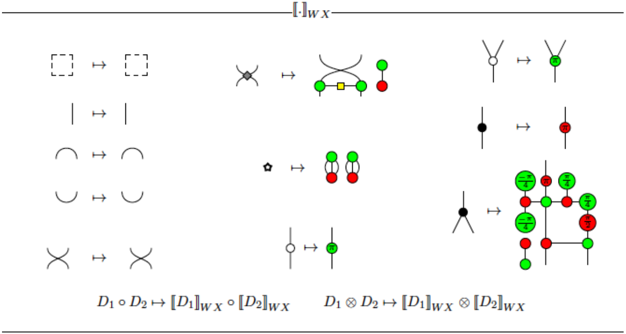
Ideas of the proof

- Translation from ZX to ZW:

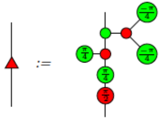


Ideas of the proof

- Translation from ZW to ZX:



- Use triangle notation in the intermediate step:



The first completeness result of unrestricted ZX-calculus

[Hadzihasanovic, Ng, Wang, LICS'18], [Ng, Wang, arXiv:1706.09877]

Complete rules I:

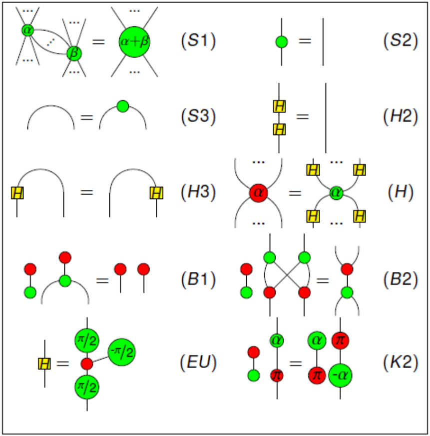


Figure: where $\alpha, \beta \in [0,\ 2\pi)$. The upside-down version and colour swapped version of these rules still hold.

The first completeness result of unrestricted ZX-calculus

Complete rules II:

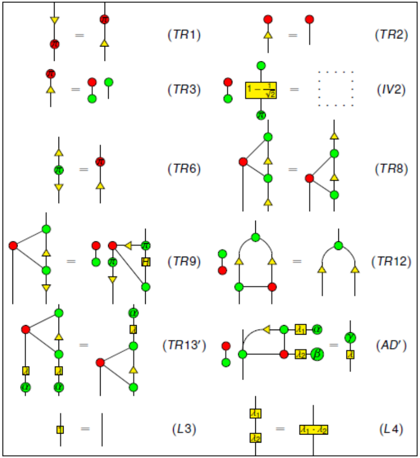


Figure: where $\lambda, \lambda_1, \lambda_2 \geq 0, \alpha, \beta, \gamma \in [0,\ 2\pi)$; in (AD′), $\lambda e^{i\gamma} = \lambda_1 e^{i\beta} + \lambda_2 e^{i\alpha}$.The upside-down version of these rules still hold.

Ideas of the proof

- Introduce new generators:

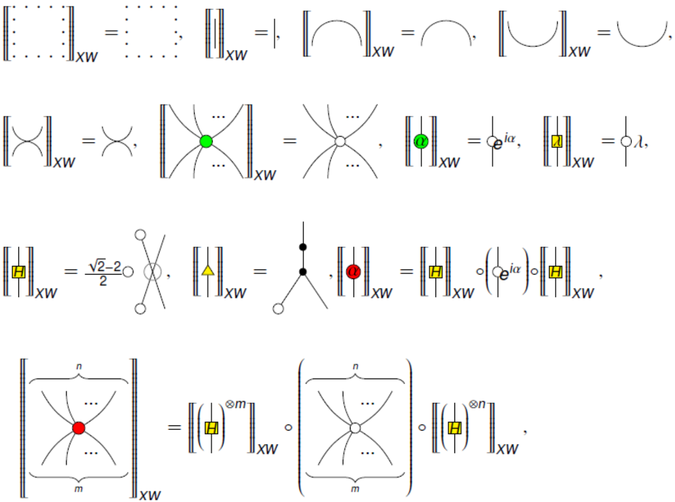
$L : 1 \to 1$ 	$T : 1 \to 1$ 

Table: New generators with $\lambda \geq 0$.

- The translation should respect the interpretations (keep it simple).

Ideas of the proof

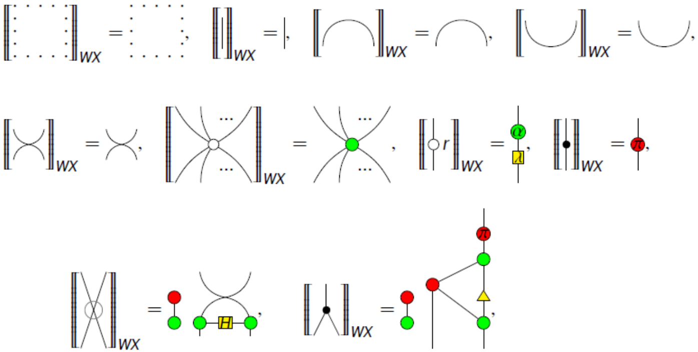
- Translation from ZW to ZX:



where $\alpha \in [0,\ 2\pi),\ \lambda \geq 0$.

Ideas of the proof

- Translation from ZX to ZW (essentially the same as in [Jeandel, Perdrix, and Vilmart, LICS'18]):



Completeness of Clifford+T ZX-calculus with new generators

[Hadzihasanovic, Ng, Wang, LICS'18], [Ng, Wang, arXiv:1801.07993]

Complete rules I:

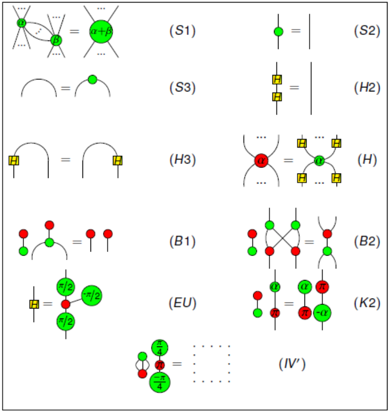


Figure: where $\alpha, \beta \in \{\frac{k\pi}{4} | k = 0, 1, \cdots, 7\}$. The upside-down version and colour swapped version of these rules still hold.

Completeness of ZX-calculus with new generators for Clifford+T QM

Complete rules II:

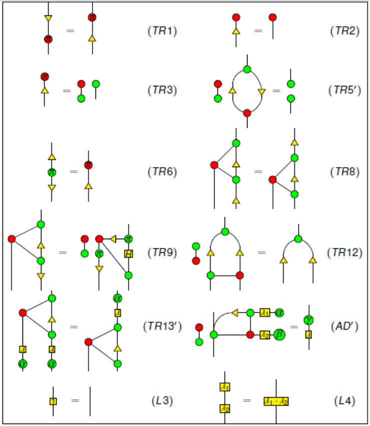


Figure: ZX_{C+T}-calculus rules with triangle and λ box, where $0 \leqslant \lambda, \lambda_1, \lambda_2 \in \mathbb{Z}[\frac{1}{2}], \alpha \in \{\frac{k\pi}{4} | k = 0, 1, \cdots, 7\}, \alpha \equiv \beta \equiv \gamma \ (mod\ \pi)$ in (AD′). The

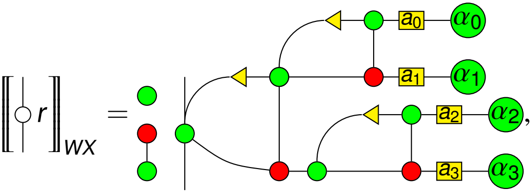
Ideas of the proof

Pretty much the same translation as the unrestricted version;

- Since we are now restricting the angles, we have to check the presentation of the theory:

Ideas of the proof

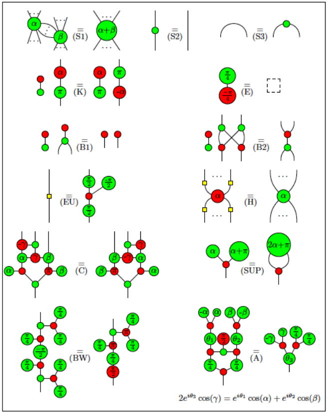
- Check that the triangle is representable with the primitive nodes;
- Since we are translating from the ZW calculus, we have to check that the Clifford+T ZX calculus is universal for matrices over a commutative ring: $\mathbb{Z}[\frac{1}{2}, e^{i\frac{\pi}{4}}]$;
- Modify the translation from ZW to ZX used in universal completeness:



where $r = a_0 e^{i\alpha_0} + a_1 e^{i\alpha_1} + a_2 e^{i\alpha_2} + a_3 e^{i\alpha_3}$, $0 \leq a_j \in \mathbb{Z}[\frac{1}{2}]$, $\alpha_j = j\frac{\pi}{4}$ or $j\frac{\pi}{4} + \pi$, $j = 0, 1, 2, 3$.

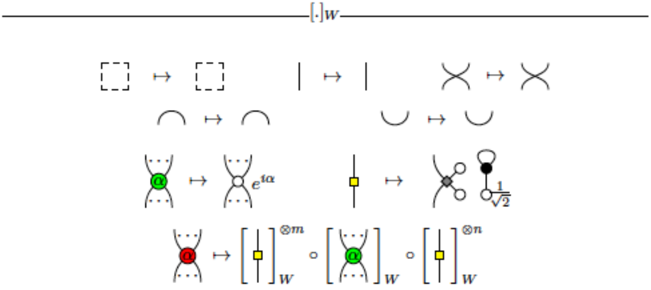
Completeness of unrestricted ZX-calculus without new generators

[Jeandel, Perdrix, and Vilmart, LICS'18]
The following ZX rules are complete for the full pure qubit QM:



Ideas of the proof

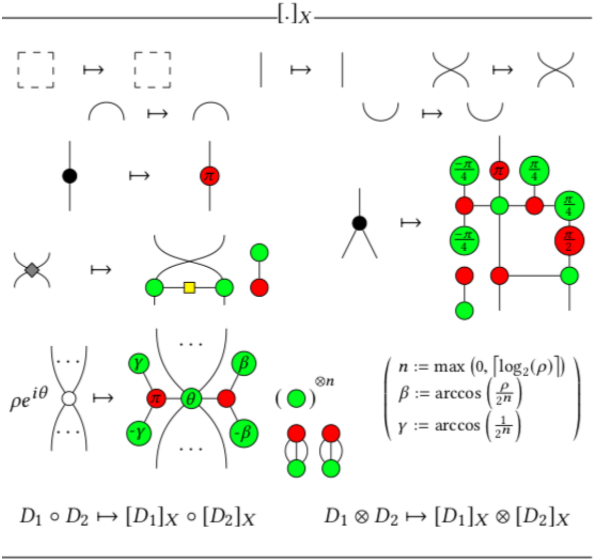
Keep it simple translation from ZX to ZW (essentially the same as used in [Hadzihasanovic, Ng, Wang, LICS'18]):



$D_1 \circ D_2 \mapsto [D_1]_W \circ [D_2]_W$ $\qquad$ $D_1 \otimes D_2 \mapsto [D_1]_W \otimes [D_2]_W$

Ideas of the proof

Translation from ZX to ZW:



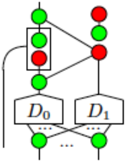
Completeness of ZX-calculus without new generators for the full pure qubit QM via normal form

- In [Jeandel, Perdrix, and Vilmart, LICS'19]: The same ZX rules as given in [Jeandel, Perdrix, and Vilmart, LICS'18] are complete for the full pure qubit QM;
- Prove via normal form rather than translation.

Normal form

- The normal form used in [Jeandel, Perdrix, and Vilmart, LICS'19] is defined recursively.
- **(Controlled scalars).** A ZX-diagram $D : 1 \to 0$ is a controlled scalar if $[\![D]\!] |0\rangle = 1$.
- **(Controlled Normal Form).** Given a set S of controlled scalars, the diagrams in controlled normal form with respect to S (S-CNF) are inductively defined as follows:

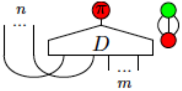
  – $\forall D \in S$, D *is in* S-CNF;

  – $\forall D_0, D_1$ *in* S-CNF,  *is in* S-CNF.

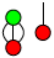
  A diagram D *in* S-CNF *is depicted*  .

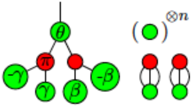
Normal form

- **(Normal Form).** Given a set S of controlled scalars, for any $n, m \in \mathbb{N}$, and any $D : 1 \to n + m$ in S-CNF, the following diagram is called a normal form with respect to S (S-NF):



- Define $\Lambda_{\mathbb{R}} : \mathbb{C} \to ZX[1, 0]$ as:

  – $\Lambda_{\mathbb{R}}(0) =$ 

  – $\forall \rho > 0, \forall \theta \in [0, 2\pi),\ \Lambda_{\mathbb{R}}(\rho e^{i\theta}) :=$ 

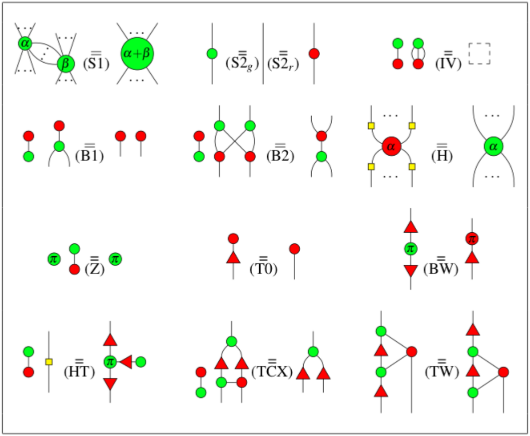
  $\begin{pmatrix} n := \max(0, \lceil \log_2(\rho) \rceil) \\ \beta := \arccos(\frac{\rho}{2^n}) \\ \gamma := \arccos(\frac{1}{2^n}) \end{pmatrix}$

  and $S_{\mathbb{R}} := \{\Lambda_{\mathbb{R}}(x) \mid x \in \mathbb{C}\}$.

- **Theorem** [Jeandel, Perdrix, and Vilmart, LICS'19] Any ZX-diagram can be put into a normal form with respect to $S_{\mathbb{R}}$, and the ZX-calculus is complete for the full pure quibt QM.

Completeness of ZX-calculus with Triangles for Toffoli-Hadamard

[Vilmart, QPL'18] The following rules are complete for Toffoli-Hadamard QM:

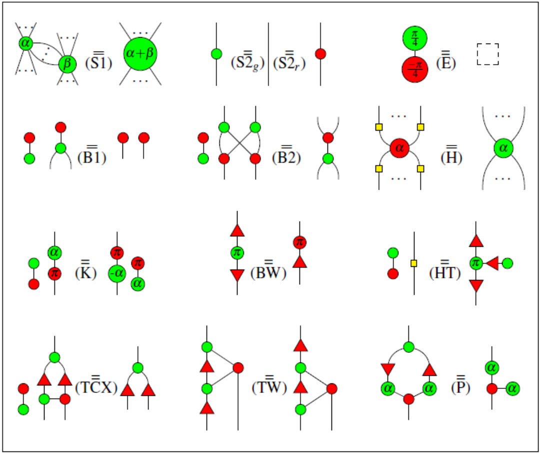


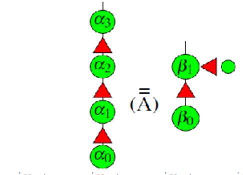
Proof again...

The same stuff again. . . (A refinement of their old translation.)

Completeness of ZX-calculus with Triangles for Toffoli-Hadamard, and beyond

Continue building to finally:





$$e^{i\alpha_3}(1+e^{i\alpha_2}(1+e^{i\alpha_1}(1+e^{i\alpha_0})))$$
$$= 2e^{i\beta_1}(1+e^{i\beta_0})$$

Translation and the triangle

The techniques used are roughly the same – translating to the ZW calculus.

- From ZW calculus perspective, we look for “missing rules”;
- Refine the missing rules, and we have our favourite ZX-calculus.

The hard part is finding our favourite presentation.

- The triangle plays an important role in many presentation;
- The triangle is in some way the black vertex, so it is not surprising that it appears everywhere – harnessing the power of the black vertex in ZW.

Completeness of the ZX-calculus for 2-qubit Clifford+T quantum circuits

[Coecke, Wang, RC'18]

The following ZX rules are complete for 2-qubit Clifford+T quantum circuits:

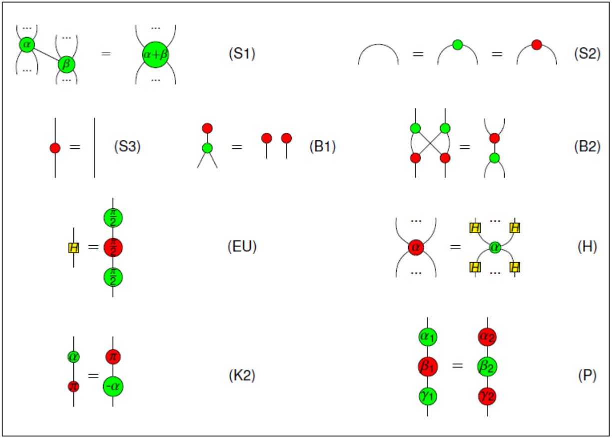


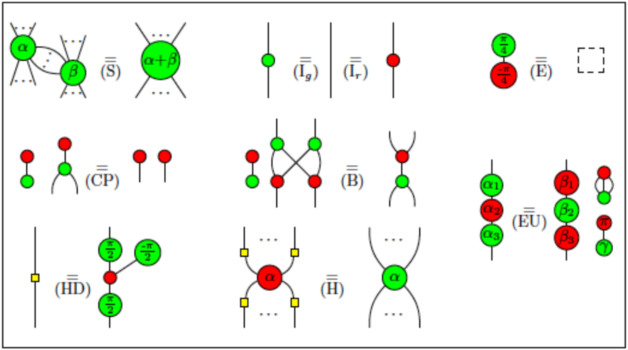
Figure: where $\alpha, \beta \in [0, 2\pi)$.

Ideas of the proof

- Translation to the complete quantum circuit relations on two qubits given in [Selinger, Bian, QPC'15].
- Interesting usage of the (P) rule.

Proof of Schröder and Zamdzhiev's conjecture on completeness

[Vilmart, LICS'19]
The following ZX rules are complete for pure qubit QM:



where (EU) is the (P) rule in [Coecke, Wang, RC'18] (with scalars).

Idea of the proof

The idea is simply to derive all the ZX-rules in some complete ZX-calculus. They chose the presentation in [Jeandel, Perdrix, and Vilmart, LICS'18].

Completeness for many different fragments...

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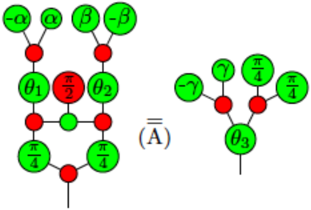
Tweaking the ZX-calculus

Some of the rules in the unrestricted ZX-calculus (in all presentation) involve horrifying calculation of angles:

- Non-linear angles

Non-linearity rules

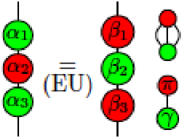
In [Jeandel, Perdrix, and Vilmart, LICS'18] and [Jeandel, Perdrix, and Vilmart, LICS'19]:



$$2e^{i\theta_3}\cos(\gamma) = e^{i\theta_1}\cos(\alpha) + e^{i\theta_2}\cos(\beta)$$

Non-linearity rules

In [Vilmart, LICS'19]:

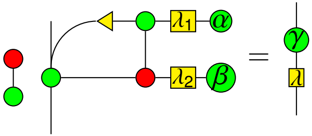


where

$$\begin{cases} x^+ := \frac{\alpha_1+\alpha_3}{2} \qquad x^- := x^+ - \alpha_3 \\ z := \cos\left(\frac{\alpha_2}{2}\right)\cos\left(x^+\right) + i\sin\left(\frac{\alpha_2}{2}\right)\cos\left(x^-\right) \\ z' := \cos\left(\frac{\alpha_2}{2}\right)\sin\left(x^+\right) - i\sin\left(\frac{\alpha_2}{2}\right)\sin\left(x^-\right) \\ \beta_1 = \arg z + \arg z \\ \beta_2 = 2\arg\left(i + \left|\frac{z}{z'}\right|\right) \\ \beta_3 = \arg z - \arg z' \\ \gamma = x^+ - \arg(z) + \frac{\alpha_2 - \beta_2}{2} \end{cases}$$

Non-linearity rules

In [Ng, Wang, LICS'18]:

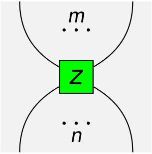


where $\lambda e^{i\gamma} = \lambda_1 e^{i\beta} + \lambda_2 e^{i\alpha}$.

Tweaking the ZX-calculus

If we look back at the translation and see how these rules came about, . . .

New green vertices



$$\mapsto \quad |0\ldots0\rangle\langle0\ldots0| + z\,|1\ldots1\rangle\langle1\ldots1|$$

where $z \in \mathbb{C}$.

Linear unrestricted ZX-calculus

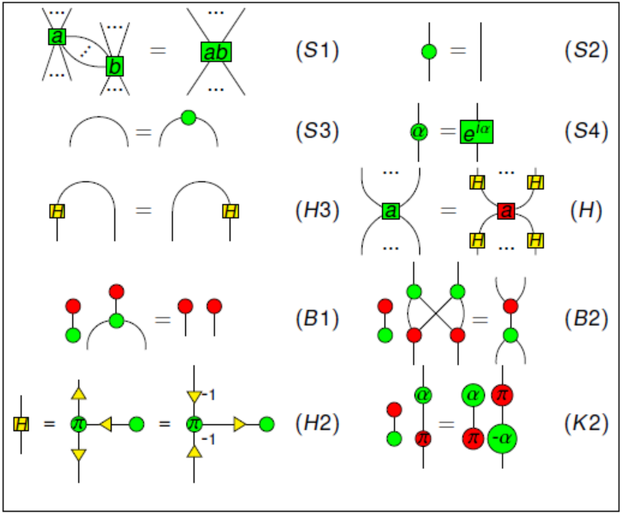
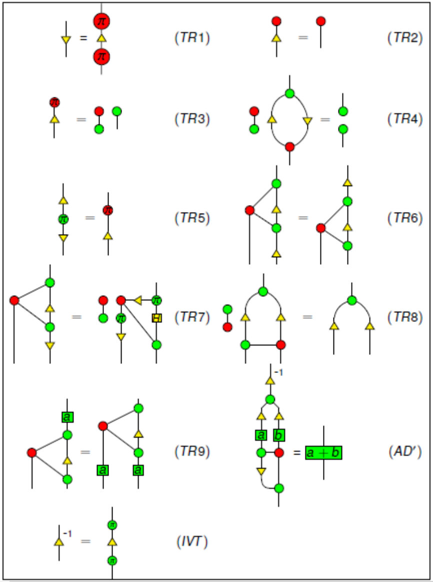
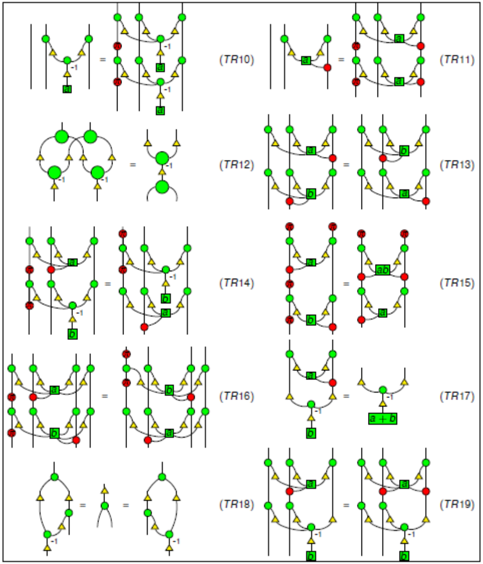


Figure: Pure linear ZX-calculus rules I, where $\alpha, \beta \in [0,\ 2\pi), a, b \in \mathbb{C}$.

Linear unrestricted ZX-calculus



Linear unrestricted ZX-calculus

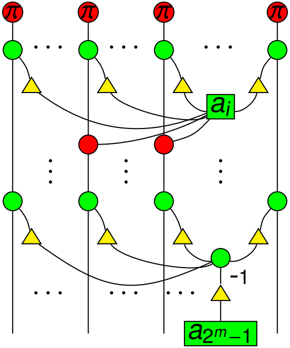


Linear unrestricted ZX-calculus

Paper coming soon! But I can give you a sneak preview (by Quanlong Wang):

Ideas of the proof

The proof is via a normal form: Any complex vector $(a_0, a_1, \cdots, a_{2^m-1})^T$ can be uniquely represented by



where a_i connects to wires by red nodes depending on i, and all possible connections are included in the normal form.

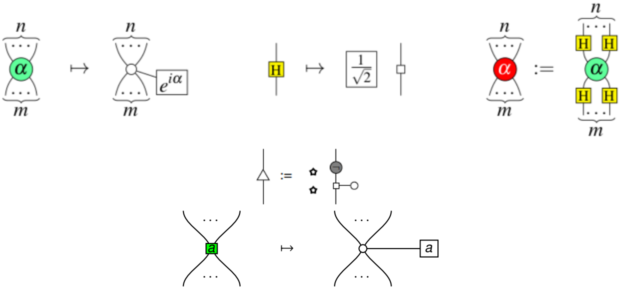
Ideas of the proof

The normal form reflects the elementary row operations:

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \xrightarrow[\text{addition}]{\text{row}} \begin{pmatrix} a_0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \xrightarrow[\text{addition}]{\text{row}} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2^m-1} \\ 1 \end{pmatrix} \xrightarrow[\text{multiplication}]{\text{row}} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{2^m-2} \\ a_{2^m-1} \end{pmatrix}$$

Relations between ZH and the linear ZX

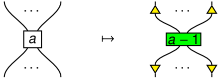
- [Backens, Kissinger, QPL'18] ZH-calculus is complete for the full pure qubit QM.
- [Backens, Kissinger, QPL'18], [de Wetering, Wolffs, arXiv:1904.07545],Translation from ZX to ZH:

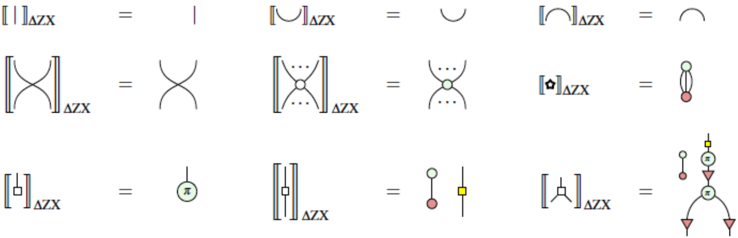


where the star has the interpretation of $\frac{1}{\sqrt{2}}$.

Relations between ZH and the linear version of ZX

- [Backens, Kissinger, QPL'18], [de Wetering, Wolffs, arXiv:1904.07545]
  - Translation from ZH to ZX:





where the last identity was also given by Quanlong Wang at the meeting of ZX 10 years, May 2018.

Further work

- Generalise the completeness result to qudits;

Further work

- Generalise the completeness result to qudits;
- A unified ZX-calculus for all the qudits in one framework;

Further work

- Generalise the completeness result to qudits;
- A unified ZX-calculus for all the qudits in one framework;
- Generalise ZX-calculus to infinite dimension (with completeness also);

Further work

- Generalise the completeness result to qudits;
- A unified ZX-calculus for all the qudits in one framework;
- Generalise ZX-calculus to infinite dimension (with completeness also);
- Useful ZX-rules for optimisation of Benchmark quantum circuits.

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Thank you!