A HoTT Quantum Equational Theory

Jennifer Paykin
Galois, Inc

Steve Zdancewic
University of Pennsylvania

QPL
June 10, 2019

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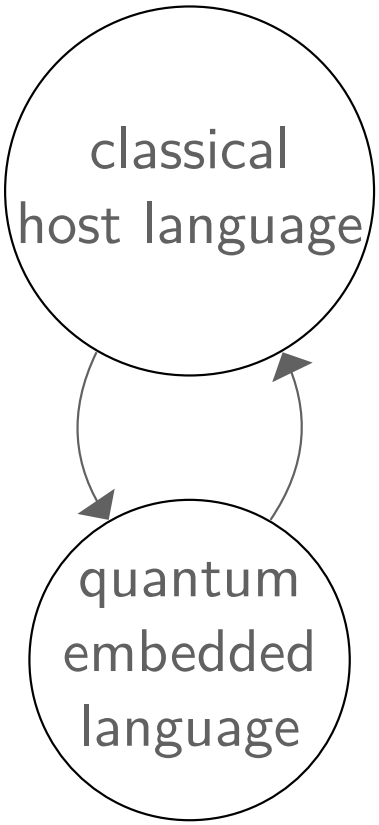
Embedded quantum languages



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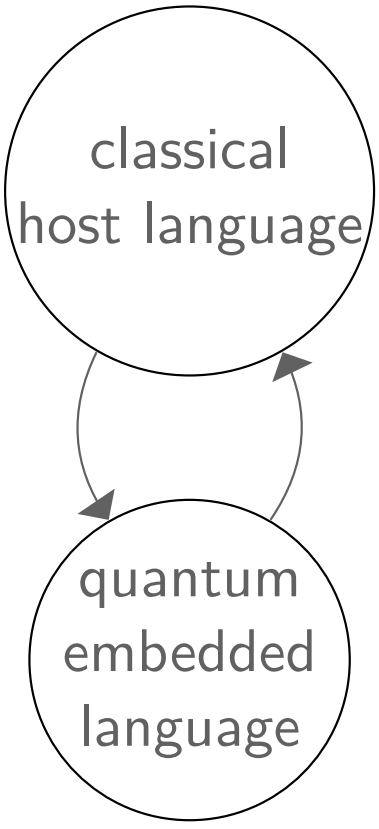

- Quantum data: qubits, unitaries, measurement
  - gate model

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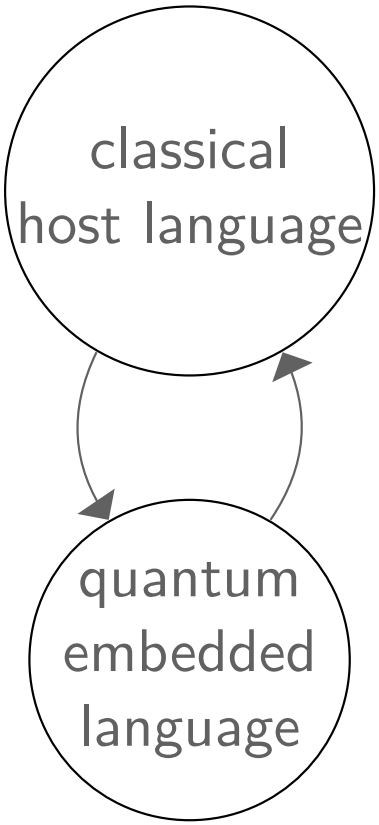
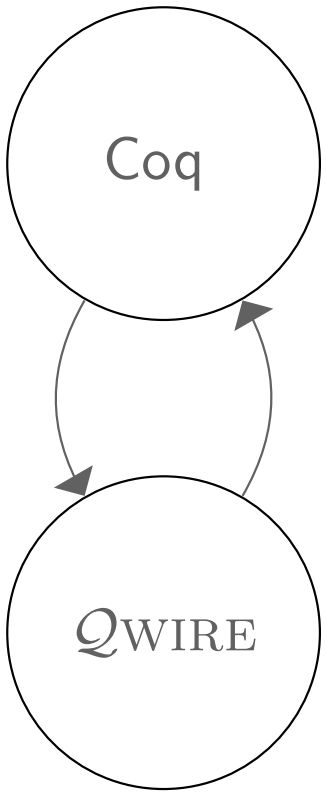


- Quantum data: qubits, unitaries, measurement
  - gate model
- Classical control: host abstractions (functions, recursion, types), tools

Reasoning about standalone quantum languages

- Denotational semantics
  - QPL [Selinger, 2004], Proto-Quipper-M [Rios and Selinger, 2018]
- Program logics
  - Ying [2011], Unruh [2019]
- Equational semantics
  - ZX calculus [Coecke and Duncan, 2011], algebraic effects [Staton, 2015]

Reasoning about embedded quantum languages



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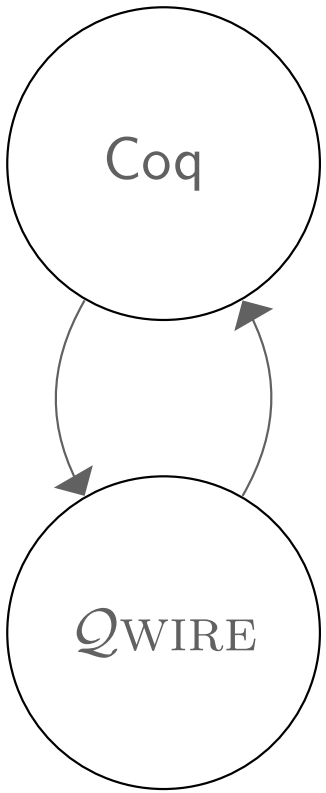

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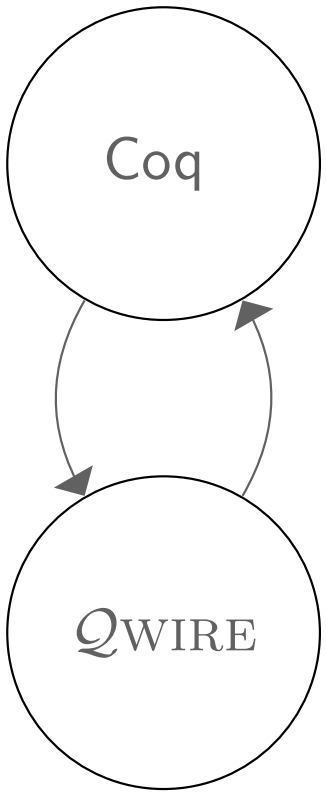

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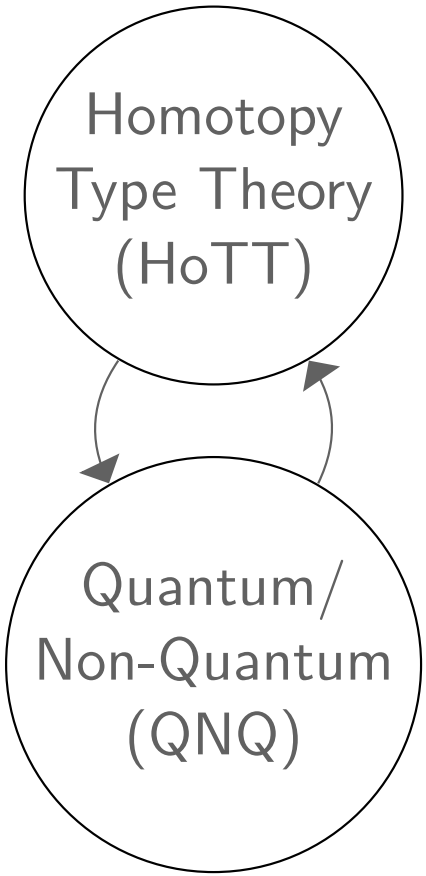
- Equational theory?

Homotopy type theory (HoTT): a type theory of equality

- Equality type: $a = b$
  - where $a, b : \alpha$



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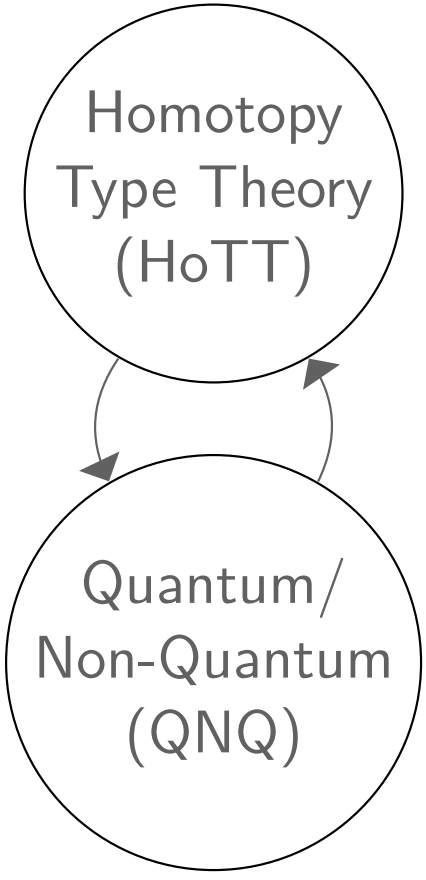

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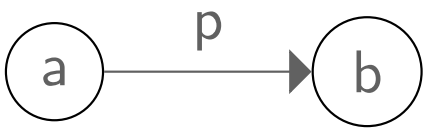

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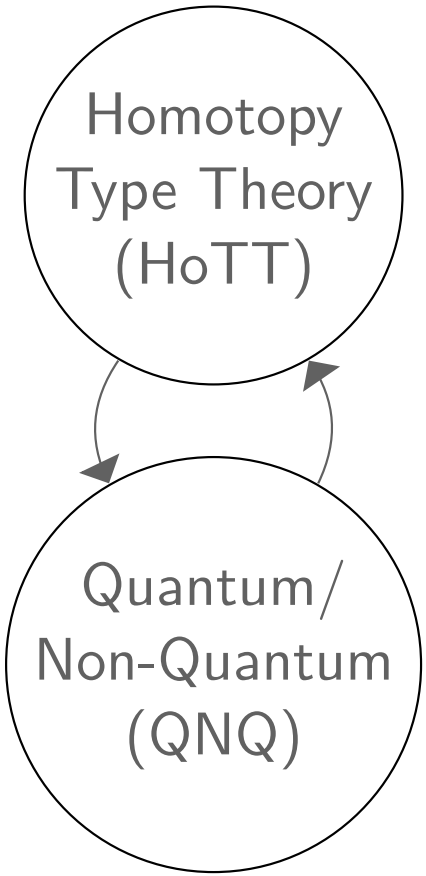

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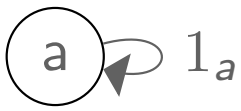

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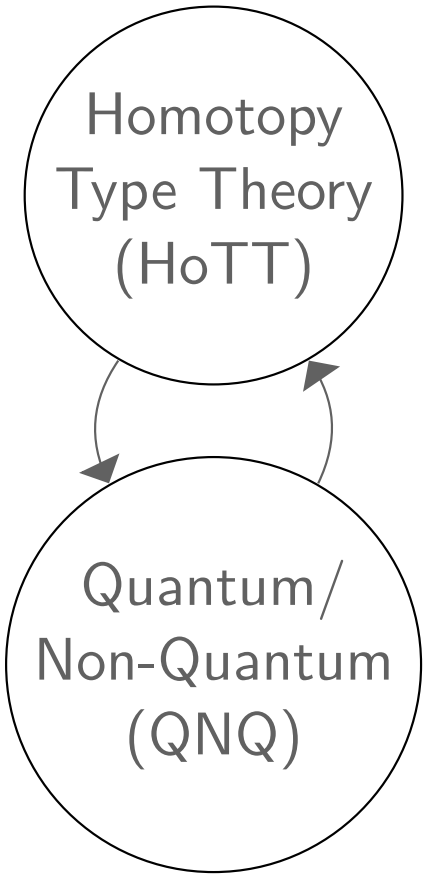

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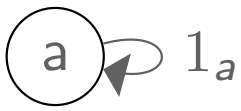

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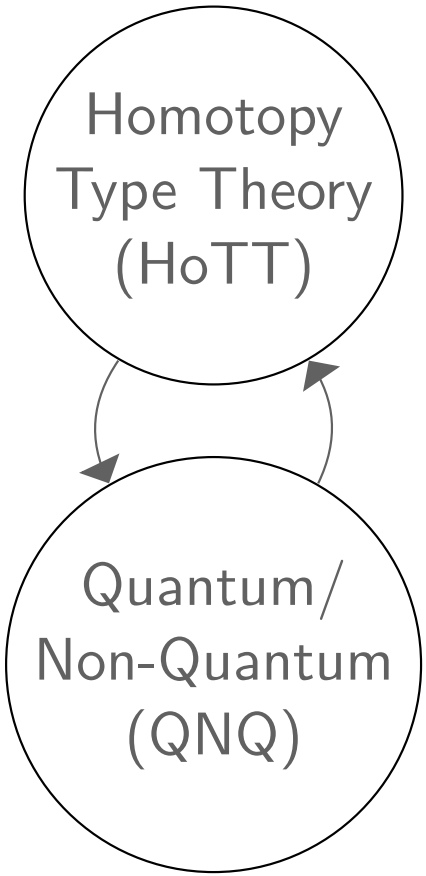

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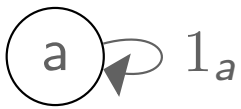

- Path induction:
  - to show $\forall(p : a = b).\ H(p)$,
  - suffices to show $\forall a.\ H(1_a)$.
- Not all paths are equal to 1_a...

Higher Inductive Type (HIT)

Definition

The quotient of a type α by a relation $R : \alpha \to \alpha \to \text{Prop}$ is a type α/R with data constructor:

$$\frac{a : \alpha}{[a]_R : \alpha/R}$$

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Note

$p : R(a, a)$ *does not imply* $[p] = 1_a$

Idea: Represent unitaries as paths

- Higher Inductive Types (HITs) use paths to encode *equivalence relations* or *groupoids*.
  - Groupoid: category where all morphisms are invertible

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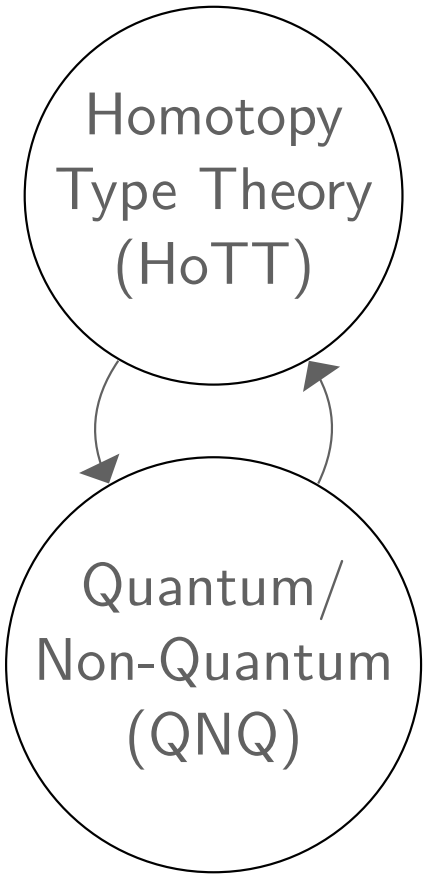
- Unitary transformations form a groupoid.

Quantum/non-quantum calculus [Paykin, 2018]



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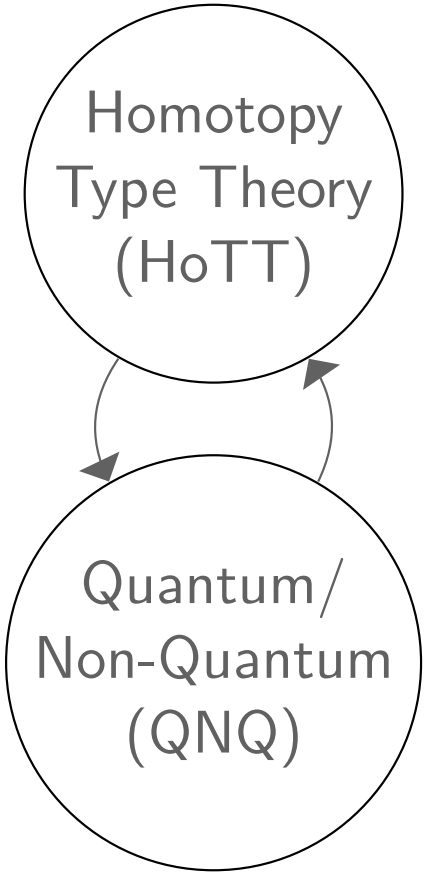

- Host terms $a : \alpha$
- Embedded quantum expressions $e : \mathsf{QExp}\ \sigma$

Quantum operators

$$\frac{b : \text{Bool}}{|b\rangle : \text{QExp Qubit}}\ \text{INIT}$$

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$$\frac{e : \mathsf{QExp}\ \mathsf{Qubit} \qquad f : \mathsf{Bool} \to \mathsf{QExp}\ \tau}{e >!\ f : \mathsf{QExp}\ \tau}\ \textsc{measure}$$

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  - $[H] \neq [X] \neq 1_{\mathsf{Qubit}}$

HoTT QNQ calculus

$$\sigma \in \mathsf{QType} = \mathsf{FinType}/\mathsf{UMatrix}$$

$$\begin{aligned} e := x \mid\ & \mathsf{let}\ x := e\ \mathsf{in}\ e' \\ \mid\ & (e_1, e_2) \mid \mathsf{let}\ (x_1, x_2) := e\ \mathsf{in}\ e' \\ \mid\ & |a\rangle \mid e >!\ f \end{aligned}$$

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- β- and $\eta-$ equivalences:

$$|b\rangle >!\ f \quad \sim_\beta \quad f\ b$$

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. . . unitaries?

Unitary application

Theorem

Let U be a unitary transformation $U : \sigma = \tau$.
$(\sigma, \tau : QType \equiv FinType/UMatrix)$

If $e : QExp\ \sigma$, there exists $U \# e : QExp\ \tau$.
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Proof.

By path induction. Base case for $1_\sigma : \sigma = \sigma$:

$$1_\sigma \# e \equiv e$$

□

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Note

$[H] \# e \neq e$ *because* $[H] \neq 1_{Qubit}$

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Equational theory for *embedded* quantum language.

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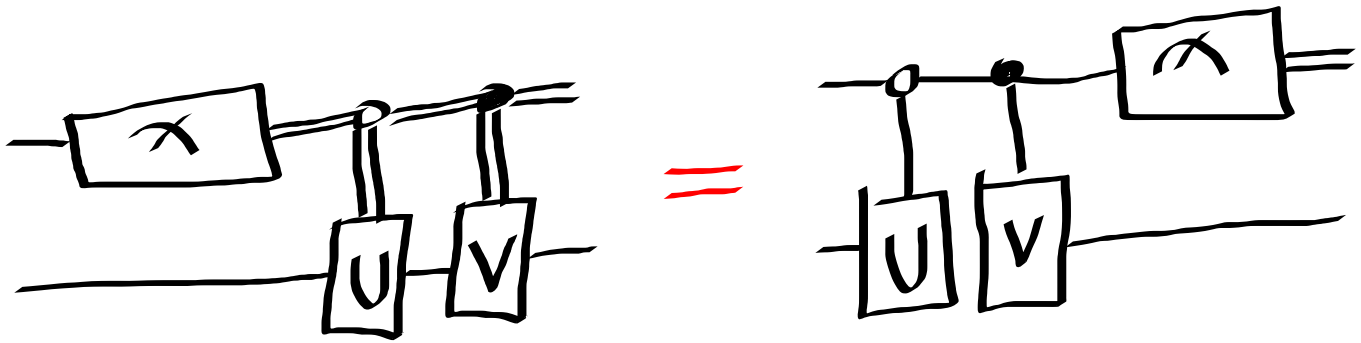
- Focus on boundary of quantum data/classical control
  - NOT equational theory for classes of unitaries
  - e.g. $XZX = -Z$ implies $[XZX] = [-Z]$

Prior work – Staton [2015]

- Equational theory for *algebra* with unitaries and classical control.



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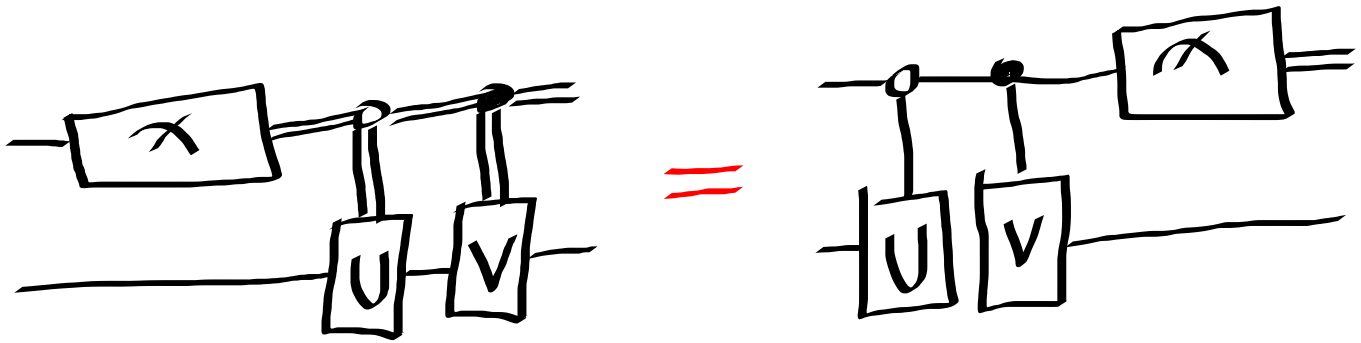
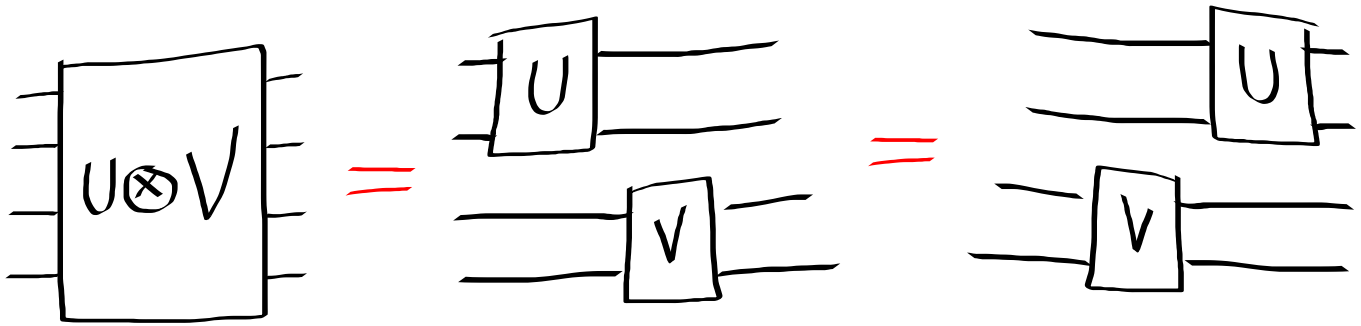


- Procedural axioms based on diagrams
  - symmetric monoidal structure



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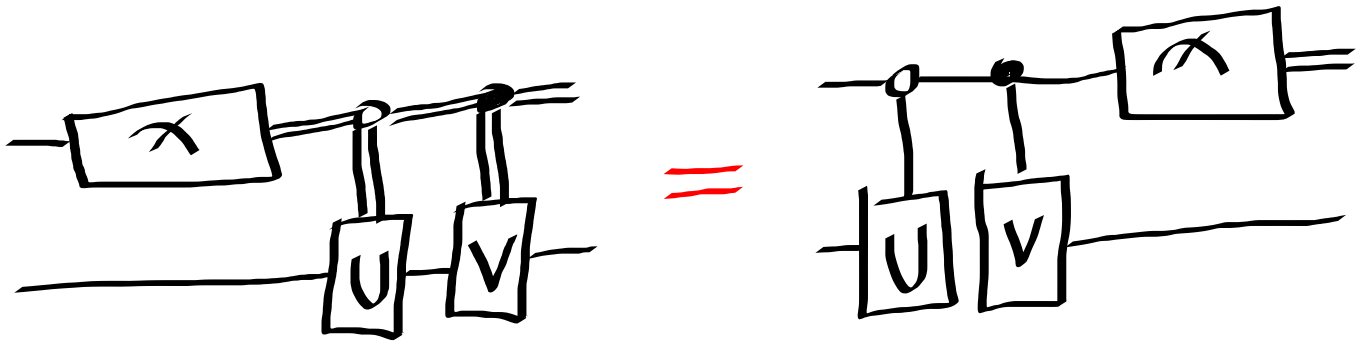

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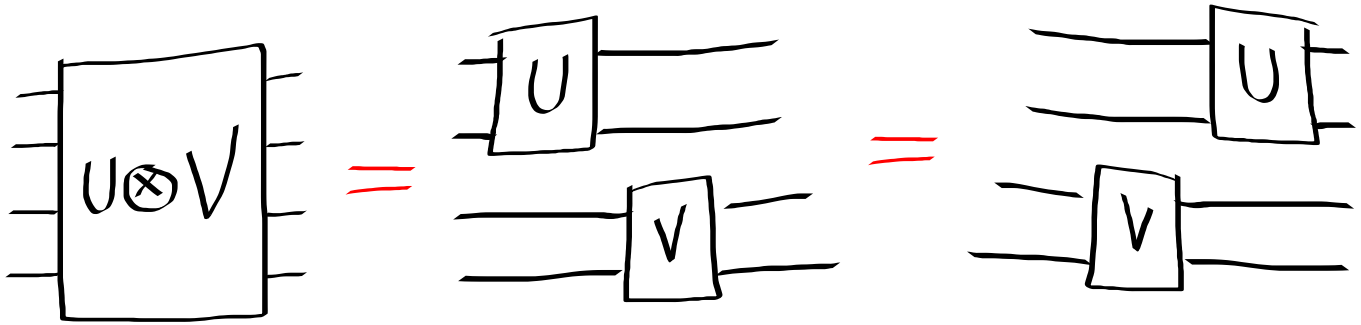

- Sound and complete with respect to C^*-algebras.
  - $d = 2$

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Equational theory for *embedded* QNQ in HoTT.

- *languages*, not algebra or diagrams
  - Variables, functions, modularity, tuples,...
- Highlight interesting axioms; fewer "procedural" axioms.
  - Paths in HoTT already yield these rules
  - Can derive many axioms for free with HoTT
- Completeness by comparison with Staton's theory.

Unitaries in the HoTT QNQ

Theorem

Let $U : \sigma = \tau$ *and* $V : \tau = \rho$ *be unitaries. Then*

$$V \# (U \# e) = (V \circ U) \# e.$$

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Proof.

By path induction on V. If $V \equiv 1_\tau$ then

$$LHS = 1_\tau \# (U \# e) = U \# e$$
$$RHS = (1_\tau \circ U) \# e = U \# e$$

□

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Define versions of †, $\otimes$, etc on paths.

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$(U_1 \otimes U_2) \# (e_1, e_2) = (U_1 \# e_1, U_2 \# e_2)$

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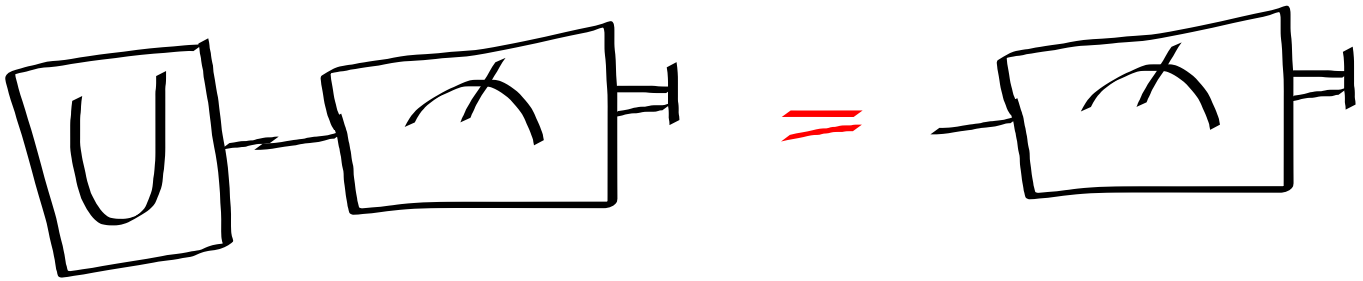
Theorem

$discard(meas(U \# e)) = discard(meas(e))$



...but not everything

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Proof.

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Similar for other "structural" unitaries:

$$\text{ASSOC} : \sigma_1 \otimes (\sigma_2 \otimes \sigma_3) = (\sigma_1 \otimes \sigma_2) \otimes \sigma_3$$

$$\text{DISTR} : \text{Qubit} \otimes \tau = \tau \oplus \tau$$

$\vdots$

Partial initialization axiom

Structural equivalence $\sigma \leftrightsquigarrow \tau$:

$$\mathsf{swap}_{X,Y} : X \times Y \to Y \times X$$
$$\mathsf{swap}_{X,Y}(x, y) = (y, x)$$

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Lift structural equivalence to unitary:

$$\widehat{\mathsf{swap}}_{\sigma,\tau} : \sigma \otimes \tau = \tau \otimes \sigma$$

such that

$$\widehat{\mathsf{swap}}_{\sigma,\tau} = [\mathsf{SWAP}_{\sigma,\tau}]$$

Partial initialization axiom

Axiom

Let $f : \sigma \leftrightsquigarrow \tau$ be a structural equivalence. Then

$$\widehat{f} \# \mathit{init}_\sigma(b) \approx \mathit{init}_\tau(f(b))$$

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$$\widehat{f} \# \mathit{init}_\sigma(b) \approx \mathit{init}_\tau(f(b))$$

Partial initialization:

$$\widehat{\mathsf{swap}_{X,Y}} \# (e_1, e_2) \approx \mathsf{swap}(e_1, e_2) = (e_2, e_1)$$

Partial measurement axiom

Axiom

Let $f : \sigma \leftrightsquigarrow \tau$. *Then:*

$$\mathit{match}\ \widehat{f}\ \#\ e\ \mathit{with}\ g \approx \mathit{match}\ e\ \mathit{with}\ (g \circ f)$$

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Partial measurement:

$$\begin{array}{l} \text{match } \widehat{\text{swap}} \ \# \ e \text{ with } \lambda(x, y).e' \\ \approx \\ \text{match } e \text{ with } \lambda(y, x).e' \end{array}$$

Results

- Two axioms:
  - structural unitaries + initialization
  - structural unitaries + measurement

[1] https://github.com/jpaykin/GroupoidQuotient

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Thanks!

A HoTT Quantum Equational Theory

Jennifer Paykin
Galois, Inc

Steve Zdancewic
University of Pennsylvania

QPL
June 10, 2019

Supported by FA9550-16-1-0082

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